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A PROCESS OF AXISYMMETRIC UNLIMITED SHOCK-FREE COMPRESSION OF A PERFECT GAS†

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Using Sidorov's ideas and analytical methods for solving the problem of the shock-free compression of a gas acted upon by a piston, a new parametric form of the solution of the equation for the self-similar velocity potential of a gas is proposed. This enables the problem of constructing the flow with an unlimited increase in the gas-dynamic parameters to be reduced to solving the Cauchy problem for a single ordinary differential equation with a bounded integration interval. The solution of the gas-dynamic equations thus obtained may be of interest in constructing the process of unlimited compression of a perfect gas, at rest at the initial instant of time inside a solid of revolution of the "plate" type, and its describes the gas flow in a certain part of the compressed volume. Asymptotic estimates of the gas-dynamic quantities are established analytically. © 2003 Elsevier Ltd. All rights reserved.

Different classes of exact solutions of the equations of gas dynamics have been obtained, which have been used to construct the processes of unlimited shock-free compression of a perfect gas, at rest at the initial instant of time inside a certain volume [1–4]. As a rule, these solutions describe the gas flow only in a certain part of the compressed volume, while in the remaining part the solutions have to be determined numerically. However, in this case one can carry out a qualitative analysis of the properties of the gas flows at points where they are constructed analytically.

The form of the initial volume of gas, which is considered in this paper, has apparently not previously been investigated. A numerical solution for a similar, though more complex, initial configuration has been obtained‡ by a different method.

The problem of finding an accurate solution which describes the gas flow for least part of the compressed volume is of interest. The general approach to the construction of a solution is the same as that proposed by Sidorov [4]. Further, by analogy with well-known methods [5, 6],§ the asymptotic properties of the exact solution constructed are investigated analytically in the case when the adiabatic exponent γ is taken from the range [1, 3]. A check that the flow obtained continuously adjoins the region where the gas is at rest is carried out by numerical integration of the ordinary differential equation, and examples of numerical calculations for $\gamma = 5/3$ and $\gamma = 7/5$.

1. FORMULATION OF THE PROBLEM

We will consider the potential flows of a uniform perfect non-viscous and non-heat conducting gas. The following equation of state corresponds to such flows

$$p = A\rho^{\gamma}, \quad A = \text{const} > 0, \quad \gamma = \text{const} > 1$$

Suppose the gas is at rest at the initial instant of time inside a solid of revolution, the qualitative form of which is shown in Fig. 1. The Ox_2 axis is the axis of symmetry and the Ox_1x_3 plane is the plane of symmetry. The initial volume is obtained from a solid of revolution with generatrix $ABB_3A_2B_2B_1A$ by removing the cone $BB_3OB_1B_2$.

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‡ Gao, Ya. M. Theoretical realisation of shock-free compression and thermonuclear combustion of two-dimensional (cylindrical and axisymmetrical) configurations of DT-gas. Preprint No. 21, M. V. Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 2000.

§See also: KUKUSHKIN, V. A., Non-self-similar compression of a tetrahedron. Institute of Mathematics and Mechanics, Urals Section of the Russian Academy of Sciences, 2001. Deposited at the All-Union Institute of Scientific and Technical Information, No. 422-V2001, 20.02.2001.

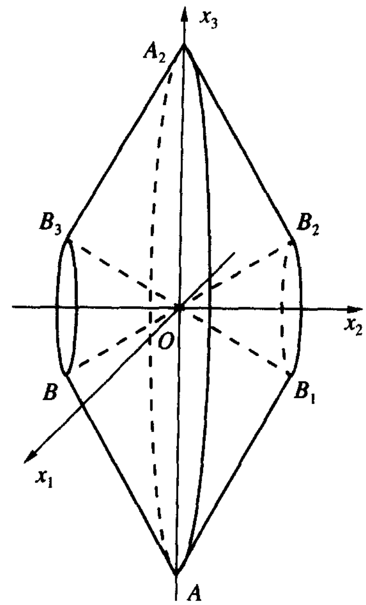

Fig. 1

We will introduce the following notation

$$r = -\sqrt{x_1^2 + x_3^2}, \quad z = x_2$$

We will consider axisymmetrical flow, in which case all the gas-dynamic quantities depend solely on the variables r and z, and it is sufficient to consider the part of the plane Ozr (Fig. 2), bounded by the conditions $r < 0$ and $z < 0$. At the initial instant of time, $t = -1$, the gas was at rest in the triangle ABO, the angle $B = \pi/2$, the section AO is a fixed impenetrable wall, and the line ABO is the initial position of a compressing piston, which at a certain instant of time $t \in (-1, 0)$ takes the form $DEFHO$. The unperturbed gas at this instant of time is in the triangle GHO.

We will change to dimensionless variables, in which the velocity of sound in the unperturbed gas is equal to unity, and choose the length of the wall BO to be equal to unity, in which case, the sonic perturbation GH arrives at the point O at the final instant of time $t = 0$. Hence, the length of the section AO is arbitrary in specifying the initial geometry of the compressed volume.

We will formulate the problem of finding an exact solution which describes the flow far from the walls BOB_1 and B_1OB_2 (Fig. 1) in the region adjoining AA_2 and bounded by the characteristic surfaces. In Fig. 2 the region DEG corresponds to this part of the compressed volume (GE is the characteristic surface). Among the solutions we will seek those which, without the formation of shock waves, lead to an unlimited increase in the density on approaching the final instant of time.

According to well-known results [3, 4], the velocity potential can be sought in the form

$$\Phi(t, z, r) = (t + 1)K - t(\Gamma(\xi, \eta) - 1/2(\xi^2 + \eta^2)), \quad K = \text{const}$$

where $\xi = z/t$, $\eta = r/t$ are self-similar variables. To find the function Γ we must solve the equation

$$\Gamma_\xi^2(\Gamma_{\xi\xi} - 1) + 2\Gamma_\xi\Gamma_\eta\Gamma_{\xi\eta} + \Gamma_\eta^2(\Gamma_{\eta\eta} - 1) - c^2(\Gamma_{\xi\xi} + \Gamma_{\eta\eta} - 3 + \eta^{-1}\Gamma_\eta) = 0 \tag{1.1}$$

where the square of the velocity of sound is equal to

$$c^2 = \kappa(\Gamma - 1/2(\Gamma_\xi^2 + \Gamma_\eta^2)), \quad \kappa = \gamma - 1 \tag{1.2}$$

In the plane of self-similar variables, the solution of Eq. (1.1) must be constructed in the unlimited region $D'G'E'$ (Fig. 3), where $G'E'$ is the characteristic and E' and D' are infinitely distant points. From the fact that the flow continuously adjoins the region at rest at the point $G' = (0, \eta_0)$ we have the equations

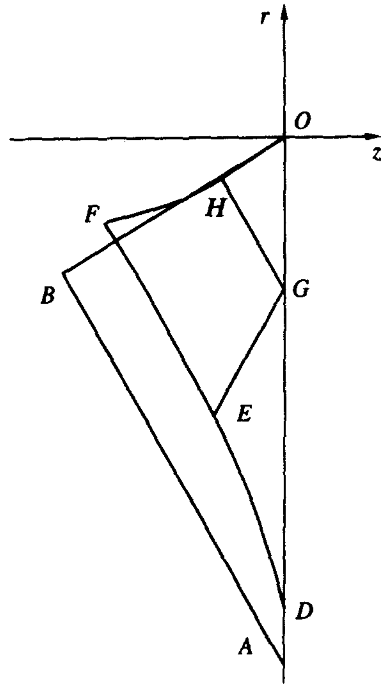

Fig. 2

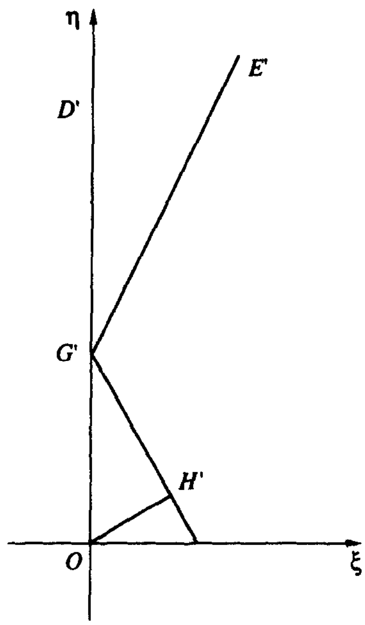

Fig. 3

$$\Gamma(0,\eta_0) = \kappa^{-1} - 1/2\eta_0^2, \quad \Gamma_\eta(0,\eta_0) = \eta_0 \tag{1.3}$$

The condition that there is no flow through the line of symmetry EG can be written in the form

$$\Gamma_\xi(0,\eta) = 0 \tag{1.4}$$

We will seek a solution of the form $\Gamma = \Gamma(\eta)$. In this case the initial problem reduces to the Cauchy problem for the ordinary differential equation

$$\Gamma_\eta^2(\Gamma_{\eta\eta}-1)-c^2(\Gamma_{\eta\eta}-3+\eta^{-1}\Gamma_\eta) = 0,\quad c^2 = \kappa(\Gamma-1/2\Gamma_\eta^2) \tag{1.5}$$

with initial data (1.3); condition (1.4) is satisfied automatically.

2. REDUCTION OF THE EQUATION FOR THE VELOCITY POTENTIAL TO A FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION

We assume the following form of the solution of Eq. (1.5)

$$\eta(a) = \eta_0\exp\left(\int_a^{a_0}\frac{da}{s(a)}\right),\quad \Gamma(a) = a\eta^2 \tag{2.1}$$

The constants η_0 and a_0 will be found later. We substitute the following expressions for the derivatives of the functions Γ

$$\begin{aligned}\frac{d\Gamma}{d\eta} &= \left(\frac{d\eta}{da}\right)^{-1}\frac{d\Gamma}{da} = (2a-s)\eta\\ \frac{d^2\Gamma}{d\eta^2} &= \left(\frac{d\eta}{da}\right)^{-1}\frac{d}{da}\left(\frac{d\Gamma}{d\eta}\right) = s\frac{ds}{da}-3s+2a\end{aligned} \tag{2.2}$$

into Eq. (1.5). After some simplifications we obtain a first-order differential equation for the function $a(s)$

$$\frac{da}{ds} = G(s,a) = \frac{s((2a-s)^2(2+\kappa)-2\kappa a)}{2(2a-3s-1)(2a-s)^2+\kappa(4(a-s)-3)((2a-s)^2-2a)} \tag{2.3}$$

and for this equation we consider the Cauchy problem with initial condition $a(0) = \tilde{a} > 0$ for different values of $\tilde{a}$.

When $s = 0$ the denominator of the right-hand side of Eq. (2.3) only vanishes when

$$a = 0, \frac{1}{2}, a_*;\quad a_* = \frac{3\kappa}{4(\kappa+1)} \tag{2.4}$$

For the remaining values of $\tilde{a}$ the equality $G(0,\tilde{a}) = 0$ is satisfied; of course, in the neighbourhood of the points $s = 0$ the following representation holds

$$a = \tilde{a}+A_1s^2+o(s^2),\quad A_1 = \text{const}\neq 0$$

Hence it follows that

$$s\sim|A_1|^{-1/2}|a-\tilde{a}|^{1/2},\quad \left|\int_{\tilde{a}}^{a_0}\frac{da}{s}\right|<\infty$$

Hence, the solution of Eq. (1.1) will only be defined for $\eta\leqslant$ const $<\infty$, and in this case this solution describes the compression of a gas to a certain finite degree of compression.

The Singular point of Eq. (2.3). The function G has a singularity at the point $(0, a_*)$ when $\tilde{a} = a_*$. We will put $s_1 = a-a_*$, $s_2 = s$, and in the numerator and denominator on the right-hand side of Eq. (2.3) we will isolate the terms that are linear in s_1 and s_2

$$\frac{ds_1}{ds_2} = G(s_2, a_*+s_1) = \frac{a_{11}s_1+a_{12}s_2+f_1(s_1,s_2)}{a_{21}s_1+a_{22}s_2+f_2(s_1,s_2)} \tag{2.5}$$

$$\frac{f_i(s_1,s_2)}{\sqrt{s_1^2+s_2^2}}\to,\quad s_1^2+s_2^2\to 0;\quad \left.\frac{\partial f_i}{\partial s_j}\right|_{s_1=s_2=0} \tag{2.6}$$

$$a_{11} = 0, \quad a_{12} = -\frac{3\kappa^2(\kappa+4)}{4(\kappa+1)^2}$$

$$a_{21} = \frac{3\kappa(2-\kappa)}{\kappa+1}, \quad a_{22} = -\frac{3\kappa(2\kappa^2+\kappa-4)}{2(\kappa+1)^2}$$

It can be verified that the following inequalities are satisfied when $\kappa < 2$

$$(a_{11}-a_{22})^2 + 4a_{12}a_{21} > 0, \quad a_{11}a_{22} - a_{12}a_{21} < 0 \tag{2.7}$$

It follows [7] from formulae (2.5)–(2.7) that the singular point is a saddle.

We will seek the solution of Eq. (2.3) for which $\lim_{s\to+0} a(s) = a_*$. Consider the linearized equation

$$\frac{ds_1}{ds_2} = \frac{a_{12}s_2}{a_{21}s_1 + a_{22}s_2}$$

for which there are two solution, which possess the property $s_1 \to 0$ when $s_2 \to 0$:

$$s_1 = k_i s_2, \quad i = 1, 2; \quad k_2 < 0 < k_1 < 1$$

where col(k_i, 1) are the eigenvectors of the matrix $A = (a_{ij})_{i,j=1,2}$ [8]. We additionally define

$$G(0, a_*) = k_1 = \frac{-2\kappa^2 - \kappa + 4 + \sqrt{-8\kappa^3 + 9\kappa^2 + 24\kappa + 16}}{4(-\kappa^2 + \kappa + 2)}$$

after which we can numerically solve Eq. (2.3).

3. THE PROPERTIES OF THE SOLUTION OF EQ. (1.5). THE RESULTS OF CALCULATIONS

Continuous adjoining of the solution to the region of rest. The initial condition $a(0) = a_*$ for Eq. (2.3) guarantees that the corresponding solution of Eqs (1.1) and (1.5) increases without limit as $\eta \to \infty$. We will determine η_0 for which conditions (1.3) are satisfied. It follows from formulae (1.3), (2.1) and (2.2) that the following equalities must be satisfied at the point $\eta = \eta_0$

$$\Gamma - \eta^2/2 = (a - 1/2)\eta^2 = \kappa^{-1}, \quad \Gamma_\eta - \eta = (2a - s - 1)\eta = 0$$

We determine numerically the value of s_0 – the root of the equation

$$\varphi(s) = 2a(s) - s - 1 = 0 \tag{3.1}$$

and we put

$$a_0 = a(s_0), \quad \eta_0 = (\kappa(a_0 - 1/2))^{-1/2}$$

The results of a calculation of the functions $a(s)$ and $\varphi(s)$ are represented in Fig. 4 for $\gamma = 5/3$ (in this case $s_0 = 0.866$, $a_0 = 0.933$ and $\eta_0 = 1.86$, curves 1) and for $\gamma = 7/5$ (in this case $s_0 = 1.78$, $a_0 = 1.39$ and $\eta_0 = 1.68$, curves 2).

The characteristic $G'E'$. The directions of the characteristics of Eq. (1.1) have the form

$$\frac{d\eta}{d\xi} = \frac{\Gamma_\xi\Gamma_\eta \pm \sqrt{\Gamma_\xi^2\Gamma_\eta^2 - (\Gamma_\xi^2 - c^2)(\Gamma_\eta^2 - c^2)}}{\Gamma_\xi^2 - c^2}$$

Taking into account the fact that, in the region $D'G'E'$ of the plane of the self-similar variables, the solution has the form $\Gamma = \Gamma(\eta)$, to find the characteristics $G'E'$, emerging from the point $G' = (0, \eta_0)$, we need to solve the Cauchy problem

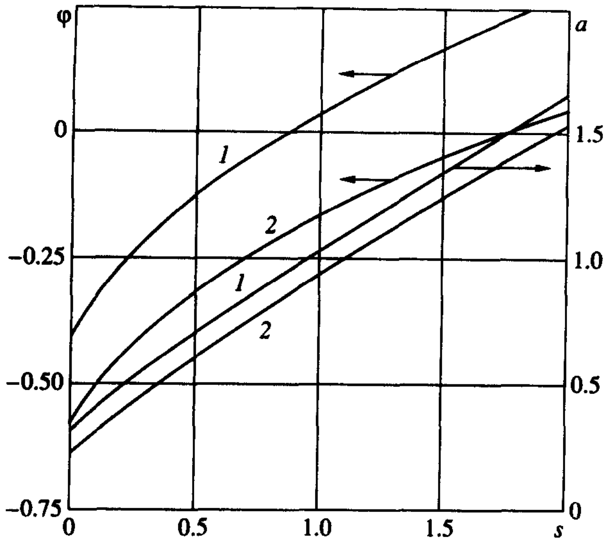

Fig. 4

$$\frac{d\eta}{d\xi} = \frac{\sqrt{\Gamma_\eta^2 - c^2}}{c}, \quad \eta(0) = \eta_0 \tag{3.2}$$

The radicand in the numerator on the right-hand side of Eq. (3.2) is equal to

$$\Gamma_\eta^2 - c^2 = (1 + 1/2\kappa)\Gamma_\eta^2 - \kappa\Gamma = \eta^2\psi(s)$$

$$\psi(s) = (1 + 1/2\kappa)(2a - s)^2 - \kappa a$$

For $\gamma = 7/5$ and $\gamma = 5/3$ we numerically verified that the function $\psi(s)$ is strictly positive. The denominator contains the velocity of sound – a positive quantity. Hence, parabolic degeneracy of Eq. (1.1) does not occur on the characteristic $G'E'$. To construct the solution over the whole volume $DFHO$ (Fig. 2) it is further necessary to solve the problem in the region $H'G'E'$ with data on the characteristics $G'H'$ and $G'E'$ (Fig. 3.)

Asymptotic estimates of the velocity, the density and the energy costs.

Assertion. Consider the region DEG at a certain instant of time $t_0 > -1$. For any gas particle from the region DEG constants R_*, V_* and D_* exist, which depend on the coordinate r of the gas particle at the instant of time $t = t_0$, such that as $t \to 0$ the following asymptotic relations hold

$$r(t) \sim R_*(-t)^{1-2a_*}, \quad v(t) \sim V_*(-t)^{-2a_*}, \quad \rho(t) \sim D_*(-t)^{-4a_*/\kappa} \tag{3.3}$$

where

$$R_{\max} \le R_* \le R_{\min} < 0, \quad 0 < V_{\min} \le V_* \le V_{\max},$$

$$0 < D_{\min} \le D_* \le D_{\max},$$

$$R_{\max}, R_{\min}, V_{\max}, V_{\min}, D_{\max}, D_{\min} = \text{const} \ne 0$$

where the constants $R_{\max}$, $V_{\max}$, $D_{\max}$ are independent of the choice of the particle and the value of t_0.

Proof. For a certain chosen particle we will determine the asymptotic form of $\eta(t)$ as $t \to -0$. We have

$$\begin{aligned} \frac{dr}{dt} &= \eta - \Gamma_\eta = (1 - 2a_*)\eta - (s + 2(a - a_*))\eta \\ \frac{d\eta}{dt} &= \frac{1}{t}\left(\frac{dr}{dt} - \eta\right) = \frac{1}{t}(-2a_* - s - 2(a - a_*))\eta \end{aligned} \tag{3.4}$$

We separate the variables in the last equation

$$\frac{dt}{t} = \int\frac{d\eta}{(-2a_* - s - 2(a-a_*))\eta} = \int\frac{d\eta}{-2a_*\eta} + f_0$$

$$f_0 = \int f_1 d\eta, \quad f_1 = \frac{f_2}{2a_*\eta(-2a_* + f_2)}, \quad f_2 = -s - 2(a-a_*)$$

We will estimate the quantities f_0, f_1 and f_2. Since, as $a \to +a_*$, the following asymptotic relations are satisfied

$$s = (a-a_*)/k_1 + o(a-a_*), \quad \eta = \eta_0 \exp\left(\int\limits_a^{a_0}\frac{da}{s}\right) \sim \text{const} \times (a-a_*)^{-k_1}$$

then

$$f_2 = -s - 2(a-a_*) \sim \text{const} \times (a-a_*) \sim \text{const} \times \eta^{-1/k_1}$$

Consequently, f_0 is a bounded function. We finally obtain

$$-t = e^{f_0}\eta^{-\frac{1}{2a_*}}, \quad \eta = e^{-f_0}(-t)^{-2a_*} \sim \text{const} \times (-t)^{-2a_*}, \quad t \to -0$$

We choose R_2 as the constant $R_{\max}$ for the point D, Fig. 2. The correctness of the second estimate of (3.3) follows from formula (3.4). The first two formulae of (3.3) are proved.

We will now prove the last formula of (3.3). We investigate the law of motion of a certain chosen particle. The equation of the z coordinate is integrated in explicit form

$$dz/dt = -\Gamma_\xi + \xi = z/t, \quad z = z_0 t, \quad z_0 = \text{const} \tag{3.5}$$

It follows from the first relation of (3.3) that

$$z(t) = o(r(t)), \quad \xi(t) = o(\eta(t)), \quad t \to 0$$

We substitute the expression for the function Γ in terms of the quantities a and η into formula (1.2)

$$c^2 = \alpha(s)\eta^2, \quad \alpha(s) = \kappa(a(s) - 1/2(2a(s) - s)^2)$$

where

$$\alpha(0) = \kappa a_*(1 - 2a_*) > 0$$

Taking into account the definition of the velocity of sound

$$c^2 = \partial p/\partial\rho, \quad p = A\rho^\gamma$$

we obtain

$$\rho \sim c^{2/\kappa} \sim \eta^{2/\kappa}$$

The assertion is proved.

The work of the piston increases the kinetic energy (E_k) and the internal energy (E_i) of the gas. Suppose V is the volume occupied by the gas and $u_{\max}$ and $\rho_{\max}$ are the greatest values of the velocity and density respectively. Then (the integrals are taken over the volume V)

$$E_k = 1/2\int u^2\rho dV \le 1/2u_{\max}^2\int\rho dV \le 1/2mu_{\max}^2$$

$$E_i = \int p dV \le \rho_{\max}^{\gamma-1}\int\rho dV \le m\rho_{\max}^{\gamma-1}$$

By analogy with the well-known approach [3] we assume that the greatest order of increase in the gas-dynamic quantities is observed in the region DEG, Fig. 2. Using the estimates of the velocity and density (the last two estimates of (3.3)), we obtain

$$E_k \sim \text{const}\, E_i \sim \text{const}(-t)^{-4a_*}, \quad t \to 0$$

Concluding remarks. The reduction of the problem of finding the self-similar velocity potential to a Cauchy problem for an ordinary differential equation with a limited integration interval enables us to say that an exact solution has been obtained for the initial problem.

The proposed approach enables one, first, to obtain a solution of Eq. (1.1) which ensures an unlimited increase in the gas-dynamic quantities in the initial problem, and second, enables one to carry out a qualitative analysis of the properties of the flow obtained.

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